Section 6.1 – Law of SINES

So far, all the triangles we've solved have had one thing in common- they have all been ***right*** triangles. However, we can use sine and cosine to solve ***oblique*** triangles too - triangles WITHOUT a right angle.

To solve an oblique triangle, you must know the measure of at least one SIDE, and any two other parts of the triangle. The possibilities are:

1) AAS 2) ASA 3) SSA 4) SAS 5) SSS

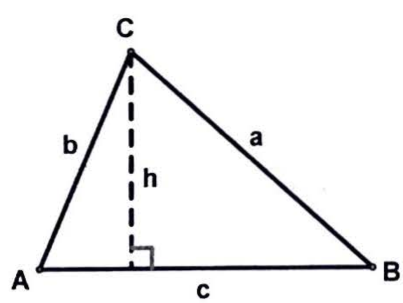
Only three of these situations can be solved with **Law of Sines** – the other two will use **Law of Cosines**. Today, we're going to discuss two of the first three.

Law of Sines

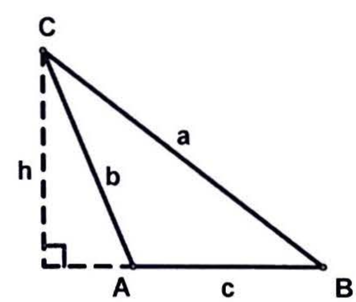
If $\triangle ABC$ has sides a, b, and c, then

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

The triangle will look like one of the two shown below:



A is acute



A is obtuse

The Law of Sines can also be written in reciprocal form:

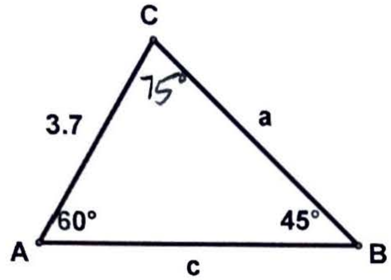
$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

Section 6.1 – Law of SINES

The AAS Case:

For the triangles below, find the remaining sides and angles

1)



m∠C= $180-60-45=75^\circ$

a= 4.53

c= 5.05

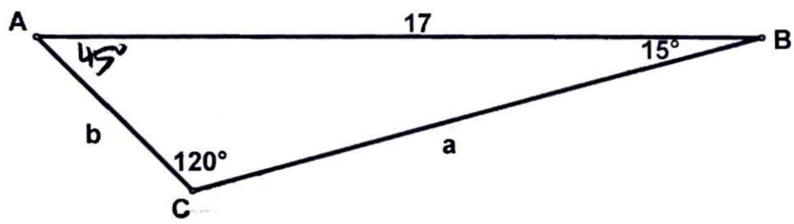
$$\frac{a}{\sin 60^\circ} = \frac{3.7}{\sin 45^\circ}$$

$$a = \frac{3.7\sin 60^\circ}{\sin 45^\circ} \approx 4.53$$

$$\frac{c}{\sin 75^\circ} = \frac{3.7}{\sin 45^\circ}$$

$$c = \frac{3.7\sin 75^\circ}{\sin 45^\circ} \approx 5.05$$

2)



m∠A= $180-120-15=45^\circ$

a= 13.88

b= 5.08

$$\frac{a}{\sin 45^\circ} = \frac{17}{\sin 120^\circ}$$

$$a = \frac{17\sin 45^\circ}{\sin 120^\circ} \approx 13.88$$

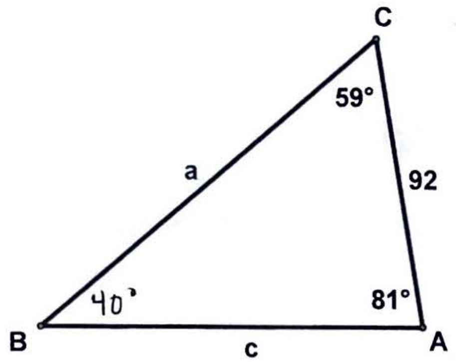
$$\frac{b}{\sin 15^\circ} = \frac{17}{\sin 120^\circ}$$

$$b = \frac{17\sin 15^\circ}{\sin 120^\circ} \approx 5.08$$

Section 6.1 – Law of SINES

The ASA case:

3)



$m\angle B = 180 - 81 - 59 = 40°$

$a = 141.36$

$c = 122.68$

$$\frac{a}{\sin 81°} = \frac{92}{\sin 40°}$$

$$a = \frac{92 \sin 81°}{\sin 40°} \approx 141.36$$

$$\frac{c}{\sin 59°} = \frac{92}{\sin 40°}$$

$$c = \frac{92 \sin 59°}{\sin 40°} \approx 122.68$$

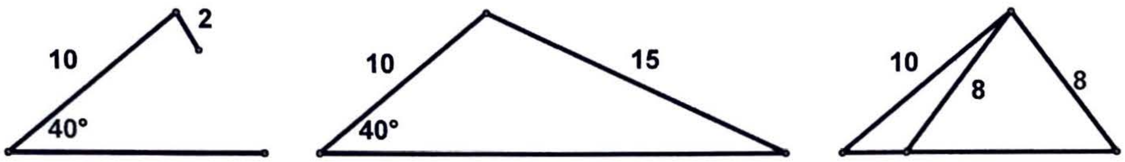
Section 6.1 – Law of SINES

The SSA case (the ambiguous case):

Why is this ambiguous?
In Geometry, you learned that you could prove that two triangles were congruent using the following methods:

SSS ASA SAS AAS

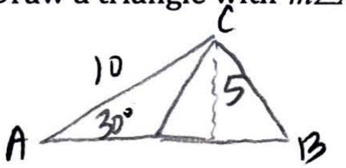
However, when you were given two sides and the NON-included angle (SSA) then, depending on the information given, you could construct 0, 1, or 2 triangles.
Here is what they look like:



0 Triangles   1 Triangle   2 Triangles

So this means that **not just one** unique triangle can necessarily be created.

How do we figure out if there are 0, 1, or 2 triangles with a SSA problem?
Draw a triangle with $m\angle A = 30°$, $b = 10$

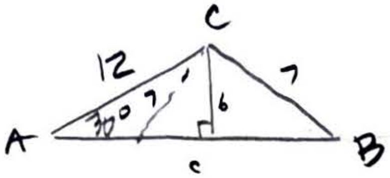


What do we know about side "a"? It is across from the $30°$ $\angle$.
If a is an altitude, then $\sin 30° = \frac{a}{10}$
$\Rightarrow a = 5$!

Now:
If $a < 5 \rightarrow$ 0 triangles
If $a > 10 \rightarrow$ 1 triangle
If $5 < a < 10 \rightarrow$ 2 triangles

Section 6.1 – Law of SINES

Ex. 1) In ΔABC, m<A = 30°, a = 7, and b = 12.
Solve the triangle for all the missing sides and angles:



$$\frac{12}{\sin B} = \frac{7}{\sin 30°}$$

$$\sin B = 12\left(\frac{\sin 30}{7}\right) = \frac{12}{14} = \frac{6}{7} \approx .8571$$

$\Rightarrow B \approx 59°$

$\Rightarrow C \approx 180 - 30 - 59 = 91°$

If $C = 91°$, then $\frac{c}{\sin 91°} = \frac{7}{\sin 30} \Rightarrow c = 14.00$

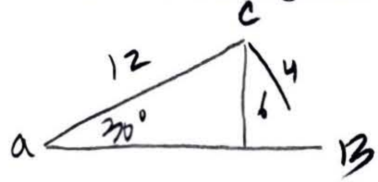
OR

$B = 180 - 59° = 121°$

$C = 29°$

If $C = 29°$, then $\frac{c}{\sin 29°} = \frac{7}{\sin 30°} \Rightarrow c = 6.79$

Ex. 2) In ΔABC, m<A = 30°, a = 4, and b = 12.
Solve the triangle for all the missing sides and angles:

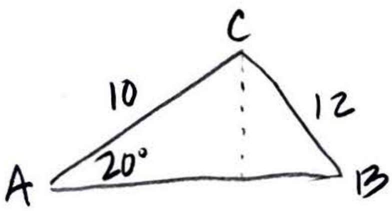


NO solution!

$$\frac{\sin B}{12} = \frac{\sin 30°}{4}$$

$\sin B = \frac{12}{8} = \frac{3}{2} > 1$ but $\sin B \leq 1$!!

Ex.3) In ΔABC, m<A = 20°, a = 12, and b = 10.
Solve the triangle for all the missing sides and angles:



$12 > 10$, so 1 solution!

$$\frac{\sin 20°}{12} = \frac{\sin B}{10}$$

$$\sin B = \frac{5 \sin 20°}{6} \approx .2850$$

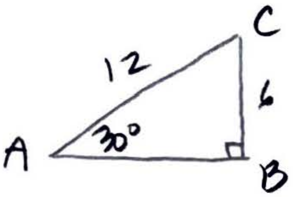
$B = 16.56°$

$C = 143.44°$

$\frac{c}{\sin 143.44°} = \frac{12}{\sin 20°} \Rightarrow c = 20.90$

Section 6.1 – Law of SINES

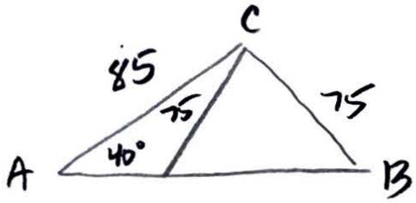
Ex.4) In ΔABC, m<A = 30°, a = 6, and b = 12.
Solve the triangle for all the missing sides and angles:



1 solution!

$$\boxed{\begin{aligned} B &= 90^\circ \\ C &= 60^\circ \\ c &= 6\sqrt{3} \approx 10.40 \end{aligned}}$$

Ex.5) In ΔABC, m<A = 40°, a = 75, and b = 85.
Solve the triangle for all the missing sides and angles:



$$\frac{\sin B}{85} = \frac{\sin 40^\circ}{75}$$

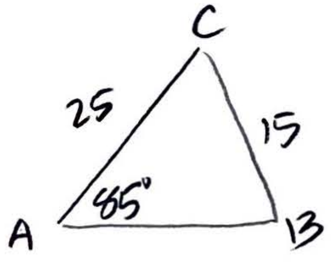
$$\sin B = \frac{85 \sin 40^\circ}{75} \approx .7285$$

$$\boxed{\begin{aligned} B &= 46.76^\circ \\ C &= 93.24^\circ \\ \frac{75}{\sin 40^\circ} &= \frac{c}{\sin 93.24^\circ} \\ c &= 116.49 \end{aligned}}$$

OR

$$\boxed{\begin{aligned} B &= 180 - 46.76 = 133.24^\circ \\ C &= 6.76^\circ \\ \frac{75}{\sin 40^\circ} &= \frac{c}{\sin 6.76^\circ} \\ c &= 13.73 \end{aligned}}$$

Ex.6) In ΔABC, m<A = 85°, a = 15, and b = 25.
Solve the triangle for all the missing sides and angles:



$$\frac{\sin B}{25} = \frac{\sin 85^\circ}{15}$$

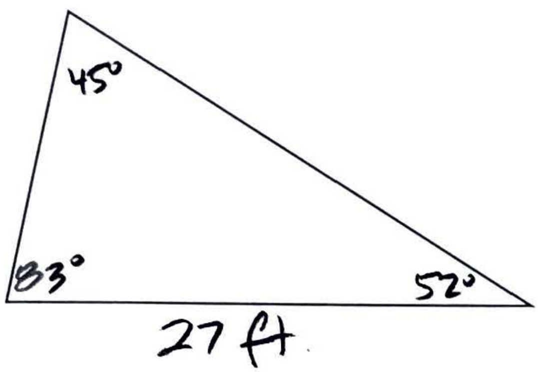
$$\sin B = \frac{25 \sin 85^\circ}{15} \approx 1.66 > 1$$

NO SOLUTION!!

Section 6.1 – Law of SINES

Law of Sines: Applications!

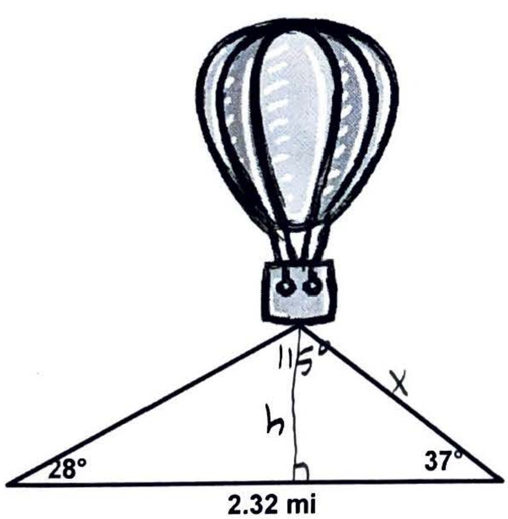
1) A telephone pole tilts AWAY from the sun at a 7° angle from the vertical, and it casts a 27-foot shadow. The angle of elevation from the tip of the shadow to the top of the pole is 52°. How tall is the pole?



$$\frac{x}{\sin 52°} = \frac{27}{\sin 45°}$$

$$x = 30.09 \text{ ft}$$

2) Observers 2.32 miles apart see a hot-air balloon directly between them but at the angles of elevation shown in the figure. Find the altitude of the balloon:



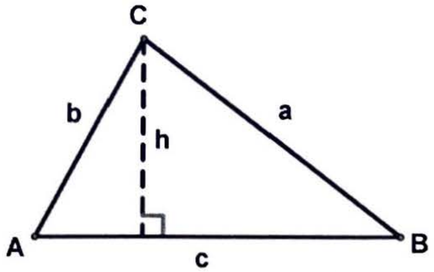
$$\frac{x}{\sin 28°} = \frac{2.32}{\sin 115}$$

$$x = 1.20 \text{ mi}$$

$$\sin 37° = \frac{h}{1.2}$$

$$h = .722 \text{ mi}$$

Area of an oblique triangle



The area of $\triangle ABC = \frac{1}{2}bh$

Now, play around and see if you can get h in terms of the sides a, b, and c.

In terms of a, h = $a \sin B$

In terms of b, h = $b \sin A$

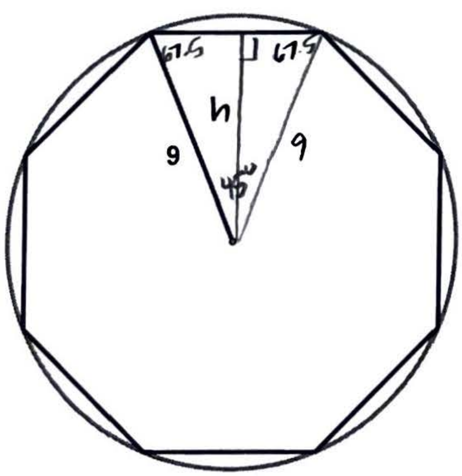
Therefore, using c as the base, the area of $\triangle ABC = \frac{1}{2} ca \sin B = \frac{1}{2} cb \sin A$

Area of an Oblique Triangle

The area of any triangle is one-half the product of the lengths of the two sides times the sine of their included angle. That is,

$$\text{Area} = \frac{1}{2} bc \sin A = \frac{1}{2} ab \sin C = \frac{1}{2} ac \sin B$$

Example: Find the AREA of a regular octagon (equiangular and equilateral) inscribed in a circle of radius 9 inches:



$\frac{360}{8} = 45$

$A = 8\left(\frac{1}{2} 9 \cdot 9 \sin 45°\right)$

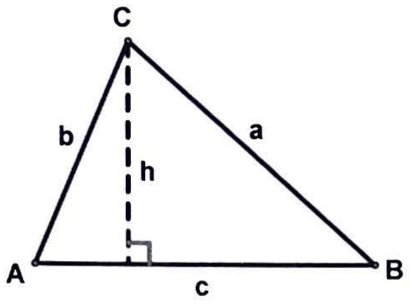
$= 8\left(\frac{81}{2} \cdot \frac{\sqrt{2}}{2}\right) = 162\sqrt{2}$

$\approx 229.11 \text{ in}^2$

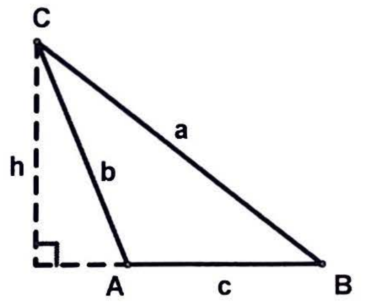
Section 6.1 – Law of SINES

In case you are curious, there is a REASON why the Law of Sines works…

Proof



A is acute



A is obtuse

Let's see why the Law of Sines is true. Considering the triangles show above, you can see that

$$\sin A = \frac{h}{b} \text{ or } h = b\sin A\text{, and}$$

$$\sin B = \frac{h}{a} \text{ or } h = a\sin B$$

From this,

$$b\sin A = a\sin B$$

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

In a similar manner (you'd need an altitude from B to side $\overline{AC}$), you should be able to show that $\frac{c}{\sin C}$ equals the other two as well.